Geometric Acoustics and Wave Theory

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GEOMETRIC WAVE THEORY

The purpose of this note is to present the theory of geometric acoustics as it emerges as a special case of the geometric theory of general linear wave propagation. The study of geometric theory of wave propagation of any type starts with a study of linear solutions in a uniform medium which are proportional to functions (generally sinusoidal) of a phase variable $\phi=\boldsymbol{\kappa}\cdot\mathbf{r}-\omega t$, with $\mathbf{r}$ a distance variable in a suitable euclidean space and $\boldsymbol{\kappa}$ a vector wave number. The study yields a relation $\omega=\Omega(\boldsymbol{\kappa})$ term a dispersion relation. If ω is real when $\boldsymbol{\kappa}$ is real, the waves are termed nondissipative. The solutions obtained are termed solutions for plane waves, the waves being planar in the $\mathbf{r}$ space. A typical example is that of gravity waves in a flat ocean of uniform depth, with the $\mathbf{r}$ space two-dimensional.

In the general geometric theory for nondissipative waves, the strict conditions above are relaxed, and an asymptotic theory in a slowly varying nonuniform medium is sought for which the local solutions are very close to those obtained for plane waves, and ω and $\boldsymbol{\kappa}$ are considered large in some relative sense. The solutions are again proportional to functions (generally sinusoidal) of a phase variable $\phi(\mathbf{r},t)$, and also to slowly varying amplitude functions. The frequency and wave number are defined by

$$\omega=-\frac{\partial\phi}{\partial t} \tag{1a}$$

$$\boldsymbol{\kappa}=\nabla\phi \tag{1b}$$

and are themselves functions of $\mathbf{r}$ and t, and satisfy

$$\frac{\partial\boldsymbol{\kappa}}{\partial t}+\nabla\omega=0 \tag{2}$$

plus the condition that $\Delta\boldsymbol{\kappa}$ is symmetric.

The dispersion relation obtained for plane waves depends upon the medium, and when applied in the asymptotic theory to the quantities defined in equation (1) gives a dispersion relation

$$\omega=\Omega(\boldsymbol{\kappa},\mathbf{r},t) \tag{3}$$

defined in an augmented space $(\boldsymbol{\kappa},\mathbf{r},t)$. The dispersion relation (3) applied to equation (2) gives a first-order partial differential equation for $\boldsymbol{\kappa}$ alone. The method of characteristics gives then, in place of equation (2), the ordinary differential equation

$$\frac{\mathrm{d}\boldsymbol{\kappa}}{\mathrm{d}t}+\nabla_{\mathbf{r}}\Omega=0 \tag{4}$$

holding along characteristics defined by

$$\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t}=\mathbf{c}=\nabla_{\kappa}\Omega \tag{5}$$

These characteristics are termed "rays." The symmetry of $\Delta\boldsymbol{\kappa}$ is used in deriving equations (4) and (5). The quantity $\mathbf{c}$ is the group velocity (ref. 1), while the quantity $\boldsymbol{\kappa}/\omega$ is the inverse phase velocity. The frequency ω obeys

$$\frac{\mathrm{d}\omega}{\mathrm{d}t}=\frac{\partial\Omega}{\partial t} \tag{6}$$

so that if the medium is steady, ω is constant along rays. The phase variable obeys the relation

$$\frac{\mathrm{d}\phi}{\mathrm{d}t}=\boldsymbol{\kappa}\cdot\mathbf{c}-\omega=\boldsymbol{\kappa}\cdot\nabla\kappa\Omega-\Omega \tag{7}$$

The rays may be parametrized by coordinates in a parameter space $\mathbf{a}$ of the same number of dimensions as the space $\mathbf{r}$. In the obvious analog with fluid mechanics in which $\mathbf{c}$ becomes the particle velocity and the rays become particle paths, the parameter space $\mathbf{a}$ becomes a Lagrangian variable space.

The slowly varying amplitude functions required for a quantitative solution may be obtained from separate equations. With the waves nondissipative this is best done through a conservation law, one in which volume integrals of an appropriate energy or wave action density are found to be conserved. In such a law the appropriate density times a measure V of an infinitesimal volume element is constant along rays. A convenient definition of V is as the determinant

$$V=|\nabla_{\mathbf{a}}\mathbf{r}| \tag{8}$$

The quantity V obeys the relation

$$\frac{d \ln V}{dt} = \nabla \cdot c \tag{9}$$

along rays.

NONDISPERSIVE WAVES

A dispersion relation is nondispersive if it is of the form

$$\omega = \kappa c_n(\mathbf{n}, r, t) \tag{10}$$

where $\boldsymbol{\kappa} = \kappa\mathbf{n}$ and $\mathbf{n}$ is a unit vector. The group velocity is then

$$\mathbf{c} = \mathbf{n} c_n + \nabla_{\mathbf{n}} c_n \tag{11}$$

with $\nabla_{\mathbf{n}}$ a gradient (normal to $\mathbf{n}$) in the unit sphere, while the inverse phase velocity is $\mathbf{n}/c_n$. Plots of $\mathbf{c}$ and $\mathbf{n}/c_n$ are found to be dual in the sense that the procedure used to go from either plot to the other is the same.

The most significant special property of nondispersive waves is that the functions of phase need not be sinusoidal but may be arbitrary. A new phase variable which is a monotonic function of the old may be introduced, if desired, and the variables ω and κ are far less significant than they are in the general case. If the medium is steady, it is convenient to define the phase so that $\omega = -1$; the phase becomes then simply a time variable measured by a fixed observer, with a suitably defined zero point.

Another special property is that the right-hand side of equation (7) is zero, so that the phase is constant along rays. This has as one consequence the result that the phase ϕ may be used as one component of the parameter space $\mathbf{a}$, and $\mathbf{a}$ replaced by $(\mathbf{a}', \phi)$.

With $\mathbf{a}$ thus replaced, equation (8) may be rewritten

$$V = c_n A_n / \omega = A_n / \kappa \tag{12}$$

where

$$A_n = \begin{vmatrix} \nabla_{\mathbf{a}'} \mathbf{r} \\ \mathbf{n} \end{vmatrix} \tag{13}$$

is a measure of the area of a ray tube formed by rays for a given value of ϕ as cut by surfaces of constant ϕ. If the right-hand side of equation (9) is divided into normal and tangential parts, we can identify

$$\frac{\mathrm{d}\ln\kappa}{\mathrm{d}t}=-\mathbf{n}\cdot\nabla\mathbf{c}\cdot\mathbf{n}, \tag{14a}$$

$$\frac{\mathrm{d}\ln A_n}{\mathrm{d}t}=\nabla_t\cdot\mathbf{c} \tag{14b}$$

Here ∇_t is the tangential gradient, normal to $\mathbf{n}$.

GEOMETRIC ACOUSTICS

A study of linear inviscid acoustic theory following the approach outlined above leads to the conclusion that its geometric theory is of the nondissipative, nondispersive type. The perturbation velocity $\mathbf{q}$ and the perturbation pressure p' are related by

$$\mathbf{n}p'=\rho a\mathbf{q} \tag{15}$$

while the dispersion relation (10) is

$$c_n(\mathbf{n},r,t)=a(\mathbf{r},t)+\mathbf{n}\cdot\mathbf{u}(\mathbf{r},t) \tag{16}$$

where a is the speed of sound and $\mathbf{u}$ is the undisturbed fluid velocity (wind). Equations (4) and (6) are replaced by equation (13) and

$$\frac{\mathrm{d}n}{\mathrm{d}t}=-\nabla_t a-(\nabla_t\mathbf{u})\cdot\mathbf{n} \tag{17}$$

$$\frac{\mathrm{d}\ln\omega}{\mathrm{d}t}=\frac{1}{c_n}\left(\frac{\partial a}{\partial t}+\frac{\partial\mathbf{u}}{\partial t}\cdot\mathbf{n}\right) \tag{18}$$

From equation (11) the group velocity is given by

$$\mathbf{c}=a\mathbf{n}+\mathbf{u} \tag{19}$$

The quantity constant along rays is $\rho q^2c_n^2A_n/\omega^2a$. The ray-tube area A_n may be obtained by a quadrature, essentially that of equation (14b), after a differential equation for the wave-front curvature $\nabla_t\mathbf{n}$ has been solved.[1] If the medium is steady, so that ω is constant along rays, the result reduces to the classical result of Blokhintsev that $\rho q^2c_n^2A_n/a$ is constant along rays.

PROPAGATION IN A STRATIFIED MEDIUM

A stratified medium is one in which the dependence of Ω on $\mathbf{r}$ and t reduces to that in one Cartesian variable, here chosen to be z. We

[1] W. D. Hayes: The Energy Invariant for Geometric Acoustics in a Moving Medium. Phys. Fluids, vol. 11, 1968, to be published.

replace $\mathbf{r}$ by $\mathbf{r}'+z\mathbf{k}$ and κ by $\kappa'+\kappa_z\mathbf{k}$. The general dispersion relation (3) takes the form

$$\omega=\Omega(\kappa',\kappa_z,z) \tag{20}$$

Equations (4) and (6) give

$$\frac{d\kappa'}{dt}=0 \qquad \frac{d\omega}{dt}=0 \tag{21}$$

A separate equation for κ_z may be given, but we may consider it given by equation (20) in terms of κ' and ω.

We use the term Snell's law for a refraction law, once integrated, for wave propagation in a stratified medium. The result, equation (21) gives directly the general Snell's law that κ' and ω are constant along rays. In particular, also, the horizontal component κ'/ω of the inverse phase velocity is constant along rays.

It is convenient to replace t by z as a basic independent variable. The ray equations (5) then take the form

$$\frac{d\mathbf{r}'}{dz}=\frac{\mathbf{c}'}{c_z}=\mathbf{K}(\kappa', \omega, z) \tag{22}$$

$$\frac{dt}{dz}=\frac{1}{c_z} \tag{23}$$

and can be integrated by quadratures to give the rays $\mathbf{r}'(z)$, $t(z)$.

When the wave propagation is nondispersive the dependence of $\mathbf{K}$ in equation (22) upon κ' and ω reduces to dependence upon the horizontal component

$$\frac{\kappa'}{\omega}=\frac{\mathbf{n}'}{c_n}=\mathbf{N}(\mathbf{a}', \phi) \tag{24}$$

of the inverse phase velocity. The volume element V may be shown to be given by

$$V=c_zA/\omega \tag{25}$$

where A is a measure of the area of a ray tube formed by rays for a given value of ϕ as cut by planes of constant z. The area A may conveniently be defined

$$A=|\nabla_{\mathbf{a}'}\mathbf{r}'| \tag{26}$$

To evaluate A we apply the operator $\Delta\mathbf{a}'$ to equation (22). The total derivative d/dz in equation (22) is a partial derivative in a

$(\mathbf{a}',z)$ space, and commutes with the operator $\nabla_{\mathbf{a}'}$. Thus we obtain

$$\frac{\mathrm{d}\nabla_{\mathbf{a}'}\mathbf{r}'}{\mathrm{d}z}=\nabla_{\mathbf{a}'}\mathbf{K}=\nabla_{\mathbf{a}'}\mathbf{N}\cdot\nabla_{\mathbf{N}}\mathbf{K} \tag{27}$$

In this equation $\nabla\mathbf{a}'\mathbf{N}$ is constant, while with $\mathbf{N}$ given $\nabla_{\mathbf{N}}\mathbf{K}$ is a function of z alone. The solution of (27) is then

$$\nabla_{\mathbf{a}'}\mathbf{r}'=(\nabla_{\mathbf{a}'}\mathbf{r}')\ z_0+\nabla_{\mathbf{a}'}\mathbf{N}\cdot\int_{z_0}\nabla_{\mathbf{N}}\mathbf{K}\ \mathrm{d}z \tag{28}$$

The determinant then gives A according to equation (26).

In the acoustic case we take $\mathbf{u}$ to be horizontal and dependent only upon z. The horizontal vector $\mathbf{K}$ may be evaluated from equation (22),

$$\mathbf{K}=\frac{a^2\mathbf{N}+(1-\mathbf{u}\cdot\mathbf{N})\mathbf{u}}{a\sqrt{(1-\mathbf{u}\cdot\mathbf{N})^2-a^2N^2}} \tag{29}$$

We define the horizontal unit vector $\mathbf{i}$ and $\mathbf{j}$ so that $\mathbf{N}=N\mathbf{i}$ and $u_t\mathbf{j}=\mathbf{u}-\mathbf{ii}\cdot\mathbf{u}$. We introduce the angle θ such that $n_z=\sin\theta$, $\mathbf{n}'=\mathbf{i}\cos\theta$, and note that $c_z=a\sin\theta$. The derivative

$$\nabla_{\mathbf{N}}\mathbf{K}=N^{-3}\frac{\cos^3\theta}{a^2\sin^3\theta}\mathbf{ii}+N^{-2}\frac{u_t\cos^3\theta}{a^2\sin^3\theta}(\mathbf{ij}+\mathbf{ji})+N^{-1}\left(\frac{\cos\theta}{\sin\theta}+\frac{u_t^2\cos^3\theta}{a^2\sin^3\theta}\right)\mathbf{jj} \tag{30}$$

is obtainable from equation (29), and permits A to be calculated through equation (28). The conserved Blokhintsev quantity is $\rho q^2c_nc_zA/a=N^{-1}\rho q^2A\sin\theta\cos\theta$.

The theory given here is equivalent to that used[2] in a recently developed computer program for calculating sonic boom pressure signatures. In such a calculation, results are first obtained using geometric acoustics and then modified for nonlinear effects.

REFERENCE

1. Whitham, G. B.: Group Velocity and Energy Propagation for Three-Dimensional Waves. Commun. Pure Appl. Math., vol. 14, 1961, pp. 675–691.

[2] W. D. Hayes et al.: Sonic Boom Propagation in a Stratified Atmosphere, with Computer Program. NASA CR, to be published.